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THE HEAT EQUATION AND MODULAR FORMS

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1. Introduction

Let G be a compact semi-simple simply connected Lie group, and let Δ be the Casimir operator on G. Then the heat equation on G is $\Delta u + \partial u/\partial t = 0$. We denote by $H(x, -t/2\pi i)$ the fundamental solution of the heat equation, and by $Z(t)$ the trace of the heat Kernel. The aim of this paper is to investigate how $H(x, t)$ behaves under the transformation $t \to -1/t$. Two main conclusions of this investigation are the following results.

Theorem 1.1. *$e^{i\pi kt/12}H(a, t) = \eta(t)^k$, where $k = \dim G$, η is the Dedekind η-function, that is, $\eta(t) = e^{i\pi t/12} \prod_{n=1}^{\infty} (1 - e^{2\pi int})$, and a is an element of G which is principal of type ρ. (For a definition of elements principal of type ρ see* [5] *or* § 5 *of this paper.)*

Theorem 1.2. *$Z(t) \sim (4\pi t)^{-(1/2)k} \operatorname{vol} G \exp(kt/24)$ is the asymptotic expansion for small t of the trace of the heat kernel.*

The first of these two results is a form of Macdonald's η-function identities. These identities can be written (see [6])

$$e^{\pi ikt/12} \Sigma d(\lambda + \rho)e^{2\pi ic(\lambda)t} = \eta(t)^k \,, \tag{1.1}$$

where the summation is over a suitable lattice which together with the other notation will be explained later. In [5] Kostant observed that this could be written as a sum over the dominant weights with a suitable weighting factor $\varepsilon(\lambda)$ as

$$e^{\pi ikt/12} \Sigma\varepsilon(\lambda)d(\lambda + \rho)e^{2\pi ic(\lambda)t} = \eta(t)_k \,. \tag{1.2}$$

Kostant identified the weighting $\varepsilon(\lambda)$ as the value of the character with highest weight λ at the point a which is principal of type ρ, that is, $\varepsilon(\lambda) = \chi_\lambda(a)$. With this result (1.2) can be interpreted in terms of the fundamental solution of the heat equation. In fact Theorem 1.1 is such an interpretation. However, our proof of Theorem 1.1 is independent of these two previous results, given by (1.1) and (1.2). Thus there is now a set of three results any two of which imply the third.

The result of Theorem 1.2 was first obtained by McKean and Singer [7] in the case of the group S^3. More recently Urakawa [10] has obtained this result

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for the other compact Lie groups. However our method of proof is different from that in [10]. For the motivation to study the function $Z(t)$, see [7].

Our method of proof in both cases is to take Fourier transforms and to use the Poisson summation formula. This process is described in the form which we shall use in § 2. Before we can apply this procedure the fundamental solution of the heat equation must be expressed as a sum over a lattice. At this point a basic difference between the two results emerges. The fundamental solution is given by

$$H(x, t) = \Sigma d(\lambda + \rho)\chi_\lambda(x)e^{2\pi i c(\lambda)t} , \tag{1.3}$$

with the sum over the dominant weights. In the case when x is a regular element of G the Weyl character formula is used to express $\chi_\lambda(x)$ as a sum over the Weyl group. We can then combine the two summations, over the dominant weights and over the Weyl group, as a single summation over a lattice. This is carried out in § 3.

In § 4 we obtain an expression for the way $H(x, t)$ behaves under the transformation $t \to -1/t$. This expression can be summarized in the following result.

Theorem 1.3.

$$H\left(x, -\frac{1}{t}\right) = \frac{e^{2\pi i B(\rho)^2/t}}{j(x)}\left(\frac{t}{i}\right)^{k/2}\frac{i^{-n}}{\text{vol } P}\Sigma d\left(\lambda - \frac{1}{2}x\right)e^{2\pi i B(\lambda-(1/2)x)^2 t} .$$

The notation is explained in § 4, and this result is proved by the methods of § 2. The proof of Theorem 1.1 is completed in § 5. With the help of Theorem 1.3 we can show that $e^{i\pi kt/12}H(a, t)$ transforms in the same way under the modular group as the kth power of the Dedekind η-function. An estimate of $e^{i\pi kt/12}H(a, t)/\eta(t)^k$ as $t \to i\infty$ then provides sufficient additional information to complete the proof of Theorem 1.1.

In § 6 we give a proof of Theorem 1.2 which is essentially as follows. If $\phi: \boldsymbol{H} \to \boldsymbol{C}$ with $\boldsymbol{H} = \{z \in \boldsymbol{C}$: imaginary part of $z > 0\}$ the upper half plane, is a modular form then the asymptotic expansion of ϕ is $\phi(t) \sim \mathrm{a}t^{-\nu}$ for suitable a and ν. This fact is well known and is a consequence of the transformation law $\phi(-1/t) = ct^\nu\phi(t)$. We shall show that $Z(-2\pi it) = e^{-i\pi kt/12}\theta(t)$ where the function $\theta(t)$ also has a transformation law. In fact the transformation law of $\theta(t)$ is more complicated than that of a modular form, but it is sufficient to give the asymptotic expansion $\theta(t) \sim a_0 t^{-(1/2)k}$. This now gives the asymptotic expansion for $Z(t) \sim ct^{-(1/2)k}\exp(kt/24)$ for some constant c. To determine the constant c we compare the first term of our expansion with the first term of the expansion of the trace of the heat kernel on a Riemannian manifold. The transformation law of $\theta(t)$ is obtained by applying the results of § 2.

Finally in § 7 we use this asymptotic expansion to compute the volume of G. It is well known that the first term in the asymptotic expansion is

$(4\pi t)^{-(1/2)k}$ vol G. By equating this with the first term of our expansion we obtain the following result.

Theorem 1.4. vol $G = (2\pi)^{l+n}$ vol $Q(R^v)/\prod_{\alpha>0} B(\alpha, \rho)$, *where* vol $Q(R^v)$ *denotes the volume of a fundamental parallelepiped of the lattice generated by the coroots with respect to the inner product induced by the Killing form.*

This result can be rewritten in the following form. Let G be a compact connected Lie group, T a fixed maximal torus in G, and $\mathfrak{g}$ and $\mathfrak{t}$ the corresponding Lie algebras. We denote by L the integer lattice, that is $2\pi L$ is the kernel of the map $\exp: \mathfrak{t} \to T$; let $\langle x, y\rangle$ be an innerproduct on $\mathfrak{g}$ invariant under the adjoint action of G. Then this induces a Riemannian structure on G. The volume of G with respect to this Riemannian structure is given by

Theorem 1.5. vol $G = C \prod$ vol S^{d_j}, *where d_j are the degrees of the generators of $H^*(G, \boldsymbol{R})$,* vol S^{d_j} *is the volume of the sphere $S^{d_j} \subset \boldsymbol{R}^{d_j+1}$ with the usual Euclidean measure, and the constant C is given by*

$$C = \text{vol}\, L \prod_{\alpha=0} \|\alpha^v\|^2 ,$$

with α^v the coroot corresponding to α and $\|\alpha^v\|^2 = \langle\alpha^v, \alpha^v\rangle$.

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2. Fourier transforms

We shall work on $\boldsymbol{R}^l$. Let $\mathscr{S}$ denote Schwartz space, that is,

$$\mathscr{S} = \{g: \boldsymbol{R}^l \to \boldsymbol{R}: |\|x\|^\gamma D^\alpha g| < C_{\gamma\alpha}\} . \tag{2.1}$$

For $g \in \mathscr{S}$ the Fourier Transform of g is

$$\hat{g}(\xi) = \int e^{2\pi i\langle\xi, x\rangle} g(x)dx , \tag{2.2}$$

and the Poisson summation formula

$$\sum_{n\in L} g(n) = \frac{1}{\text{vol}\, L} \sum_{m\in L^*} \hat{g}(m) , \tag{2.3}$$

where L is a lattice in $\boldsymbol{R}^l$, and $L^* = \{m \in \boldsymbol{R}^l: \langle m, n\rangle \in \boldsymbol{Z}$ for all $n \in L\}$ is the dual of L. By vol L we denote the volume of a fundamental cell of L taken with respect to the innerproduct $\langle x, y\rangle$ used in (2.2) and normalized so that volume of the standard integer lattice is vol $(\boldsymbol{Z}^l) = 1$. These facts can be found in [9].

Let $g(x) = e^{i\pi\|x\|^2 t}$ where $\|x\|^2 = \langle x, x\rangle$. Then an elementary calculation shows that

(2.4) $$\hat{g}(\xi) = (t/i)^{-(1/2)l} e^{-i\pi\|\xi\|^2/t} .$$

However we require the Fourier transform of the function $g(\lambda) = f(\lambda)e^{i\pi\|\lambda\|^2 t}$ when f is a homogeneous polynomial of degree 2n. This is given by

Proposition 2.1. *Let f be a homogeneous polynomial of degree $2n$ and $g(\lambda) = f(\lambda)e^{i\pi\|\lambda\|^2 t}$. Then the Fourier transform of g is*

$$g(\xi) = (t/i)^{-(4n+l)/2} i^{-2n} \{\mathscr{H} f(\xi)\} e^{-i\pi\|\xi\|^2/t} ,$$

where $\mathscr{H}$ is the operator given by

$$\mathscr{H} f(\xi) = \sum_{r=0}^{\infty} \Delta^r f(\xi) \frac{t^r}{(-4\pi i)^r r!} .$$

and Δ is the Laplacian associated to the norm $\|\xi\|^2$.

Remark. Formally we can write $\mathscr{H} = \exp(-t\Delta/4\pi i)$.

Proof of Proposition 2.1. The following result is well known. If $f(\lambda) = \Sigma_\alpha a_\alpha \lambda^\alpha$, using multi-indices, and $g(\lambda) = f(\lambda)e^{i\pi\|\lambda\|^2 t}$, then

(2.5) $$\hat{g}(\xi) = (2\pi i)^{-2n} \left(\frac{t}{i}\right)^{-(1/2)l} \Sigma_\alpha a_\alpha \left(\frac{1}{i} \frac{\partial}{\partial \xi}\right)^\alpha e^{-i\pi\|\xi\|^2/t} .$$

For convenience of notation let $u = -2\pi i/t$. We need to introduce the Hermite polynomials with parameter u and to give some results about them.

Definition. The kth Hermite polynomial with parameter u is $h_k(x, u) = e^{-(1/2)ux^2}(d/dx)^k e^{(1/2)ux^2}$.

With this definition we obtain from (2.5)

(2.6) $$\hat{g}(\xi) = (2\pi i)^{-2n} \left(\frac{t}{i}\right)^{-(1/2)l} i^{-2n} \Sigma_\alpha a_\alpha h_\alpha(\xi, u) e^{-i\pi\|\xi\|^2/t} .$$

Here we have used the assumption that f is homogeneous of degree $2n$, and the multi-index notation is interpreted so that if $\alpha = (\alpha_1, \cdots, \alpha_l)$, then $h_\alpha(\xi, u) = h_{\alpha_1}(\xi_1, u) \cdots h_{\alpha_l}(\xi_l, u)$.

Now it is a fact that

(2.7) $$h_k(x, u) = u^k \sum_{r=0}^{\infty} \left(\frac{d}{dx}\right)^{2r} x^k \frac{t^r}{(-4\pi i)^r r!} .$$

This fact is clearly true when $k = 0$ and $k = 1$. To prove it we claim that both sides satisfy the recurrence relation

(2.8) $$h_{k+1}(x, u) = \frac{d}{dx} h_k(x, u) + uxh_k(x, u) ,$$

which can be checked by a direct claculation.

To complete the proof of Proposition 2.1 we proceed as follows. First change coordinates so that $\|\xi\|^2$ is diagonal. Next use (2.7) on each variable in turn in (2.6). Observe that the series in (2.7) is an exponential series and so has the usual additive property. The result now follows upon changing coordinates back to the original ones.

3. The fundamental solution of the heat equation

We need some facts about Lie groups. The proofs of these can be found in [2]. Let G be a compact semi-simple simply connected Lie group. Fix a maximal torus $T \subset G$, and let $\mathfrak{t}$ and $\mathfrak{g}$ denote their Lie algebras respectively. Let $P \subset t^*$ be the lattice of weights of G, and choose a fundamental Weyl chamber $D \subset t^*$. Then the set $P \cap D$ is the set of dominant weights. There is a one-one correspondence between the finite dimensional irreducible representations of G and the points in $P \cap D$. We denote the innerproduct on $\mathfrak{t}^*$ induced from the Killing form by $B(x, y)$. Let $b(x, y) = \mathrm{tr}\,(adxady)$ be the Killing form on $\mathfrak{t}$. Then b is negative definite since G is semi-simple and compact. The innerproduct $B(x, y)$ is the innerproduct on $\mathfrak{t}^*$ induced by $-b(x, y)$, where we have choosen the sign conventions to make $B(x, y)$ positive definite. The following result is well known; see [4].

Lemma 3.1. *The eigenvalues of Δ are $c(\lambda)$ for $\lambda \in P \cap D$ where* $c(\lambda) = B(\lambda + \rho)^2 - B(\rho)^2$.

Here ρ is half the sum of the positive roots and $B(x)^2 = B(x, x)$.

Let $d(\lambda) = \prod_{\alpha>0} B(\lambda, \alpha) / \prod_{\alpha>0} B(\rho, \alpha)$ where the products are over the set of positive roots of G. Then if V_λ is the representation space corresponding to λ, the Weyl dimension formula gives

$$\dim V_\lambda = d(\lambda + \rho)\ . \tag{3.1}$$

It is known that the multiplicity of the eigenvalue $c(\lambda)$ is $(\dim V_\lambda)^2$.

The kernel of the heat equation is known [1] to be

$$K(x, y, t) = \Sigma_j \phi_j(x)\phi_j(y)e^{2\pi i\lambda_j t}\ , \tag{3.2}$$

where the summation is over a complete orthonormal set of eigenfunctions $\{\phi_j\}$ with λ_j the eigenvalue corresponding to ϕ_j. Notice that we have made an elementary change of variables replacing the usual equation $\Delta u + \partial u/\partial t = 0$ by $\Delta u - (1/2\pi i)(\partial u/\partial t) = 0$. In the case of a Lie group we can express the kernel as a convolution kernel which we shall call the fundamental solution. Using the fact that the eigenfunctions are the matrix coefficients of irreducible representations we can show that the fundamental solution is

$$H(x, t) = \sum_{\alpha \in P \cap D} (\dim V_\lambda)\chi_\lambda(x)e^{2\pi i c(\lambda) t}\ . \tag{3.3}$$

Let $j(h) = \sum_{\omega\in W} (-1)^\omega e^{2\pi i B(\omega(\rho),h)}$. Then the Weyl character formula gives

$$\chi_\lambda(x) = \sum_{\omega\in W} (-1)^\omega e^{2\pi i B(\omega(\lambda+\rho),x)}/j(x) . \tag{3.4}$$

Since H is a class function, it is sufficient to work with H restricted to the maximal torus T. Now lift $H(x, t)$ to the Lie algebra $\mathfrak{t}$. By identifying $\mathfrak{t}$ with $\mathfrak{t}^*$ using the innerproduct $B(x, y)$ we find that $H(x, t)$ defined for $x \in \mathfrak{t}^*$ is given by

$$H(x,t) = \frac{e^{-2\pi i B(\rho)^2 t}}{j(x)} \sum_{\lambda\in P\cap D} \sum_{\omega\in W} d(\lambda+\rho)(-1)^\omega e^{2\pi i B(\omega(\lambda+\rho),x)} e^{2\pi i B(\lambda+\rho)^2 t} , \tag{3.5}$$

where we have substituted the value $c(\lambda) = B(\lambda+\rho)^2 - B(\rho)^2$.

We can use following three facts to write H as a sum over the lattice P.

(1) The map $\lambda \to \lambda + \rho$ maps $P \cap D$ onto $P \cap D^\circ$, the lattice points in the interior of D.

(2) The polynomial $d(\lambda)$ vanishes for λ in the walls of D.

(3) There are the invariance properties $d(\sigma\lambda) = (-1)^\sigma d(\lambda)$ and $B(\sigma\lambda)^2 = B(\lambda)^2$ for $\sigma \in W$, that is, $d(\lambda)$ is skew invariant and $B(\lambda)^2$ is invariant under the Weyl group.

With these facts (3.5) can be expressed as

$$H(x,t) = \frac{e^{-2\pi i B(\rho)^2 t}}{j(x)} \sum_{\lambda\in P} d(\lambda) e^{2\pi i (B(\lambda)^2 t + B(\lambda,x))} . \tag{3.6}$$

This formula only holds for h a regular element. When h is not regular we need a different procedure. For the trace of the heat kernel instead of taking $H(1, t)$ we shall take $Z(t) = \Sigma e^{-\lambda t}$ where $\{\lambda\}$ is the set of eigenvalues of Δ counting multiplicities. Thus we can express $Z(t)$ in the following form:

$$Z(t) = \sum_{\lambda\in P\cap D} d(\lambda+\rho)^2 e^{-B(\lambda+\rho)^2 t + B(\rho)^2 t} . \tag{3.7}$$

Thus we can put $Z(-2\pi i t) = e^{-2\pi i B(\rho)^2 t}\theta(t)$ where $\theta(t)$ is given by

$$\theta(t) = \sum_{\lambda\in P\cap D} d(\lambda+\rho)^2 e^{2\pi i B(\lambda+\rho)^2 t} . \tag{3.8}$$

Consider the map $P \to P$ defined by $\lambda \to \lambda + \rho$. Under this map $P \cap D$ goes onto $P \cap D^\circ$. Now $d(\lambda) = 0$ if λ is in one of the walls of D. Both $d(\lambda)^2$ and $B(\lambda)^2$ are invariant under the action of the Weyl group W. If $\lambda \in P$ and $d(\lambda) \neq 0$, then the orbit of λ under W has a unique element in D. So under the action of the Weyl group, (3.8) becomes

(3.9) $$\theta(t) = \frac{1}{|W|} \sum_{\lambda \in P} d(\lambda)^2 e^{2\pi i B(\lambda)^2 t} .$$

Here $|W|$ denotes the order of the Weyl group.

4. The inversion formula

In this section we shall give an expression for the way $H(x, t)$ transforms under $t \to -1/t$. This expression is

Theorem 4.1.

$$H\left(x, -\frac{1}{t}\right) = \frac{e^{2\pi i B(\rho)^2/t}}{j(x)} \left(\frac{t}{i}\right)^{k/2} \frac{i^{-n}}{\text{vol } P} \sum_{\lambda \in (1/2)mQ^v} d(\lambda - \tfrac{1}{2}x) e^{2\pi i B(\lambda - (1/2)x)^2 t} .$$

The notation is as follows. Let $\{\alpha\}$ be the root system of G with respect to T. Then there is the opposite root system $\{\alpha^v\}$. The map $\alpha \to \alpha^v$ is characterized by the two conditions:

(1) $\alpha(\alpha^v) = 2$,

(2) $x - x(\alpha^v)$ is a root if x is a root.

The root system $\{\alpha\} \subset \mathfrak{t}^*$ and $\{\alpha^v\} \subset \mathfrak{t}$. The Killing form can also be defined by $B(x, y) = \Sigma \alpha(x)\alpha(y)$, and we define a new innerproduct $(x|y) = \Sigma x(\alpha^v) y(\alpha^v)$ on $\mathfrak{t}^*$.

Definition. $m = \|\alpha_0 + \rho\|^2 - \|\rho\|^2$ with $\|x\|^2 = (x|x)$, α_0 is the highest root, and $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$ is half the sum of the positive roots.

The number m satisfies $(x|y) = mB(x, y)$.

Now we introduce the lattice Q^v. Let $Q(R^v)$ be the lattice in $\mathfrak{t}$ generated by $\{\alpha^v\}$. Then Q^v is given by

$$Q^v = \{x \in \mathfrak{t}^* : (x|y) = y(\beta^v) \text{ for some } \beta^v \in Q(R^v) \text{ and all } y \in \mathfrak{t}^*\} .$$

It is a fact that $\frac{1}{2}mQ^v \subset P$ and $P^* = \frac{1}{2}mQ^v$, where P is the lattice of weights, and P^* is the dual lattice with respect to $2B(x, y)$, that is, $P^* = \{x \in \mathfrak{t}^* : 2B(x, y) \in Z \text{ for all } y \in P\}$. These facts can be found in [2].

To prove Theorem 4.1 we start with (3.6) in the form

(4.1) $$H(x, t) = \frac{e^{-2\pi i (B(\rho)^2 t + B(x)^2/4t)}}{j(x)} \sum_{\lambda \in P} d(\lambda) e^{2\pi i B(\lambda + x/2t)^2 t} ,$$

and let

(4.2) $$\theta(x, t) = \sum_{\lambda \in P} d(\lambda) e^{2\pi i B(\lambda + x/2t)^2 t} .$$

With respect to the innerproduct $2B(x, y)$ the lattices P and $\frac{1}{2}mQ^v$ are dual. Now using elementary facts about the Fourier transform and Poisson summation formula we have

$$\theta(x, t) = \left(\frac{t}{i}\right)^{-(1/2)(2n+l)} \frac{i^{-n}}{\operatorname{vol} P} \sum_{\xi \in (1/2)mQ^v} d\left(\frac{\partial}{\partial \xi}\right) e^{4\pi i B(\xi, h/2t) - 2\pi i B(\xi)^2/t} .$$

Completing the square in the exponent gives

$$(4.3) \quad \theta(x, t) = \left(\frac{t}{i}\right)^{-(1/2)k} \frac{i^{-n}}{\operatorname{vol} P} e^{i\pi B(x)^2/2t} \sum_{\xi \in (1/2)mQ^v} d\left(\frac{\partial}{\partial \xi}\right) e^{-2\pi i B(\xi - (1/2)x)^2/t} .$$

with

$$k = 2n + l = \dim G .$$

Now we proceed as in the proof of proposition 2.1. This introduces the expression $\mathscr{H} d$ where $\mathscr{H}$ is the operator $\mathscr{H} = \Sigma t^r \Delta^r / [(-4\pi i)^r r!]$.

Proposition 4.2. $\mathscr{H} d = d$.

Proof. This follows at once from the fact that d is harmonic. To prove that d is harmonic observe that since $d(\lambda) = \prod_{\alpha>0} B(\lambda, \alpha)/B(\rho, \alpha)$, d has the property that it is skew invariant, and if f is another skew invariant polynomial then d divides f. Since Δ is invariant, Δd is again skew invariant and so d divides Δd. But the degree of $\Delta d \leq$ (degree of d) $-$ 2, so we have $\Delta d = 0$.

We can now use the same methods as in Proposition 2.1 to give

$$\theta(x ,t) = \left(\frac{t}{i}\right)^{-k/t} \frac{i^{-n}}{\operatorname{vol} P} e^{i\pi B(x)^2/2t} \sum_{\lambda \in (1/2)mQ^v} d\left(\lambda - \frac{1}{2}x\right) e^{2\pi i B(\lambda - (1/2)x)^2/t} .$$

Substituting this into (4.1) gives the result in Theorem 4.1.

5. Macdonald's η-function identities

In this section we shall prove Theorem 1.1. To do this we shall evaluate $H(x, t)$ at $x = a$ where a is principal of type ρ, and then show that $e^{i\pi kt/12} H(a, t)$ has the same transformation laws under the modular group as $\eta(t)^k$. A simple estimate of $e^{i\pi kt/12} H(a, t)/\eta(t)^k$ as $t \to i\infty$ gives sufficient information to show that these two functions are equal. We start by describing the element a.

Let $\beta\colon \mathfrak{t} \to \mathfrak{t}^*$ be the isomorphism defined by $\lambda(x) = B(\lambda, \beta(x))$ for $x \in \mathfrak{t}$ and $\lambda \in \mathfrak{t}^*$. Then $a \in G$ is principal of type ρ if a is conjugate to $\exp(\beta^{-1}(2\rho))$, since $H(x, t)$ only depends on the conjugacy class of x it is sufficient to define a upto conjugation. For more details about a see [5]. Thus when we lift H to $\mathfrak{t}^*$, the element corresponding to a is 2ρ.

The modular group is generated by $T\colon t \to t + 1$ and $S\colon t \to -1/t$. Under these two transformations we have the following transformations of $\eta(t)$:

$$(5.1) \qquad \eta(t + 1) = e^{i\pi/12} \eta(t) , \qquad \eta(-1/t) = (t/i)^{1/2} \eta(t) .$$

Since $H(x, t + 1) = H(x, t)$, it is immediately verified that $e^{i\pi kt/12} H(a, t)$ has the same transformation under T as $\eta(t)^k$.

In the next section we shall prove the result $B(\rho)^2 = k/24$. If we use this and Theorem 4.1 we obtain the expression

$$e^{-(i\pi k/12)t}H\left(2\rho, -\frac{1}{t}\right) = \frac{(t/i)^{(1/2)k}i^{-n}(-1)^{\sigma}}{j(2\rho)\ \mathrm{vol}\ P} \sum_{\lambda\in(1/2)mQ^v} d(\lambda+\rho)e^{2\pi iB(\lambda+\rho)^2t}, \tag{5.2}$$

where σ is the element of the Weyl group such that $\sigma(\rho) = -\rho$ (in [2] σ is denoted by ω_0).

On the other hand from (3.6) we have

$$e^{i\pi kt/12}H(2\rho, t) = \frac{1}{j(2\rho)} \sum_{\lambda\in P} d(\lambda)e^{2\pi i(B(\lambda)^2t+B(\lambda,2\rho))}. \tag{5.3}$$

This summation can be written as a summation over $\frac{1}{2}mQ^v$ and one over a set of coset representatives of $P/(\frac{1}{2}mQ^v)$.

$$e^{i\pi kt/12}H(2\rho, t) = \frac{1}{j(2\rho)} \sum_{\xi\in P/((1/2)mQ^v)} \left\{e^{4\pi iB(\xi,\rho)} \sum_{\lambda\in(1/2)mQ^v} d(\lambda+\xi)e^{2\pi iB(\lambda+\xi)^2t}\right\}. \tag{5.4}$$

Under the action of the Weyl group $P/(\frac{1}{2}mQ^v)$ becomes the disjoint union of orbits which have the following two properties, both of which can be found in [11].

(1) There is a unique orbit in $P/(\frac{1}{2}mQ^v)$ on which W acts transitively, and this orbit contains a coset with representative ρ.

(2) If $\mu\in P$ defines a coset $\bar{\mu}$ such that the stabilizer of $\bar{\mu}$ under W is non trivial, then there is $s\in W$ such that $(-1)^s = -1$ and $s\bar{\mu} = \bar{\mu}$.

From property (2) it follows that if $\mu\in P$ is such that $\bar{\mu}$ has a non trivial stabilizer, then

$$\sum_{\lambda\in(1/2)mQ^v} d(\lambda+\mu)e^{2\pi iB(\lambda+\mu)^2t} = 0.$$

Thus we have the expression

$$e^{i\pi kt/12}H(2\rho, t) = \frac{1}{j(2\rho)}\left(\sum_{\omega\in W}(-1)^{\omega}e^{4\pi iB(\omega\rho,\rho)} \sum_{\lambda\in(1/2)mQ^v} d(\lambda+\rho)e^{2\pi iB(\lambda+\rho)^2t}\right). \tag{5.5}$$

Comparing (5.2) and (5.5) gives

$$e^{-i\pi k/12t}H(2\rho, -1/t) = c(t/i)^{(1/2)k}e^{i\pi kt/12}H(2\rho, t), \tag{5.6}$$

where $c = i^{-n}(-1)^{\sigma}/(\text{vol } P \sum_{\omega \in W} (-1)^{\omega} e^{4\pi i B(\omega\rho, \rho)})$.

To complete the proof we make some estimates as $t \to i\infty$. Let $q = e^{2\pi i t}$ Then as $t \to i\infty$, $q \to 0$. Now

$$\sum_{\lambda \in (1/2)mQ^v} d(\lambda + \rho) e^{2\pi i B(\lambda+\rho)^2 t} = d(\rho) q^{k/24} + O(q^{1+k/24}) ,$$

and

$$\eta(t) = q^{1/24} + O(q^{1+1/24}) ,$$

which gives

$$\lim_{t \to i\infty} \left(\frac{e^{i\pi k t/12} H(2\rho, t)}{\eta(t)^k} \right) = \frac{1}{j(2\rho)} \left(\sum_{\omega \in W} (-1)^{\omega} e^{4\pi i B(\omega\rho, \rho)} \right) d(\rho) .$$

However by definition $j(2\rho) = \sum_{\omega \in W} (-1)^{\omega} e^{4\pi i B(\omega\rho, \rho)}$ and $d(\rho) = 1$, so

$$\lim_{t \to i\infty} \left(\frac{e^{i\pi k t/12} H(2\rho, t)}{\eta(t)^k} \right) = 1 . \tag{5.7}$$

As in [11] this now implies that $c = 1$, and $e^{i\pi k t/12} H(2\rho, t)/\eta(t)^k$ is a bounded entire function and so is constant. The limit (5.7) gives that this constant is one, which completes the proof.

6. The asymptotic expansion

We start with the results of § 3, that is,

$$Z(-2\pi i t) = e^{-2\pi i B(\rho)^2 t} \theta(t) . \tag{6.1}$$

Thus to complete the proof of Theorem 1.2 we need to establish the two facts:

$$(1) \quad B(\rho)^2 = k/24 , \qquad (2) \quad \theta(t) \sim a_0 t^{-(1/2)k} , \tag{6.2}$$

where k is the dimension of G. The first of these is the "strange formula" of Freudenthal and de Vries (see [3]), and it follows from the second fact.

Lemma 6.1. *Suppose that $\theta(t) \sim a_0 t^{-(1/2)k}$. Then $B(\rho)^2 = k/24$.*

Proof. The assumption $\theta(t) \sim a_0 t^{-(1/2)k}$ and (6.1) give the asymptotic expansion for $Z(t)$ as

$$Z(t) \sim C t^{-(1/2)k} e^{B(\rho)^2 t} , \tag{6.3}$$

for $C = a_0/(-2\pi i)^{-(1/2)k}$. Now in [7] the asymptotic expansion is given as

$$Z(t) \sim (4\pi t)^{-(1/2)k} \text{vol } G(1 + tR/6 + O(t^2)) , \tag{6.4}$$

where R is the scalar curvature. Comparing the first two terms in the expansions

(6.3) and (6.4) gives the values of the constant C and the formula $B(\rho)^2 = R/6$. From [8] we have that the curvature tensor is

$$R(X, Y)Z = \tfrac{1}{4}[[X, Y], Z] \,. \tag{6.5}$$

Thus the scalar curvature is $R = k/4$.

To complete the proof of Theorem 1.2 we have only to establish the second fact of (6.2). We now prove this.

Lemma 6.2. $\theta(t) \sim a_0 t^{-(1/2)k}$.

Proof. We need to establish a formula which gives the behaviour of $\theta(t)$ under the transformation $t \to -1/t$. To do this we shall use the Fourier transform and Poisson summation formula. Let $g(\lambda) = d(\lambda)^2 e^{i\pi B(\lambda)^2 t}$. Then from Proposition 2.1.

$$\hat{g}(\xi) = (t/i)^{-1/2(4n+l)} i^{-2n} (\mathscr{H} d(\xi)^2) e^{-i\pi B(\xi)^2} \,. \tag{6.6}$$

Now $k = 2n + l$, that is, the dimension of G is twice the number of positive roots plus the rank of G. For convenience we put $C(t) = (t/i)^{-(1/2)k} i^{-2n}/\mathrm{vol}\, P$. The Poisson summation formula (2.3) gives

$$\sum_{\lambda \in P} d(\lambda)^2 e^{i\pi B(\lambda)^2 t} = C(t) \sum_{\xi \in P^*} (t/i)^{-n} \{\mathscr{H} d(\xi)^2\} e^{-i\pi B(\xi)^2/t} \,. \tag{6.7}$$

Replacing t by $-1/t$ we obtain the formula

$$\theta(-1/2t) = \frac{C(-1/t)}{|W|} \sum_{\xi \in P^*} \left(\frac{t}{i}\right)^n \{\mathscr{H} - d(\xi)^2\} e^{i\pi B(\xi)^2 t} \,. \tag{6.8}$$

Let $q = e^{i\pi t}$. Then there is the expansion "about infinity"

$$\sum_{\xi \in P^*} (t/i)^n \{\mathscr{H} - d(\xi)^2\} e^{i\pi B(\xi)^2 t} = b_0(t) + b_1(t) q + \cdots \,, \tag{6.9}$$

where $b_r(t) = (t/i)^n \Sigma \mathscr{H} - d(\lambda)^2$ with the sum over those λ such that $B(\lambda)^2 = r$. We have used the fact that for $\xi \in P^*$ then $B(\xi)^2 \in Z$. This follows from the results in [2]. Let R^v be the opposite root system to that of G, and let $Q(R^v)$ be the lattice of roots generated by R^v. Then $P^* = mQ(R^v)$. Here m is the number such that $(x \mid y) = mB(x, y)$ where $(x \mid y)$ is the innerproduct induced from R^v in the same way that $B(x, y)$ is induced from the root system of G.

The next result we shall need is that $b_0(t)$ is independent of t. This follows from the fact that the degree of $d(\lambda)^2$ is $2n$. Hence

$$\mathscr{H} - d(\lambda)^2 = \Delta^n d(\lambda)^2 t^{-n}/[(4\pi i)^n n!] = g(\lambda) \tag{6.10}$$

for some g satisfying $g(0) = 0$, where $\mathscr{H} -$ is just the operator $\mathscr{H}$ when t has been replaced by $-1/t$. Now $b_r(t)$ is a polynomial in t so as $t \to i\infty$, $b_r(t)q_r \to 0$ for $r \neq 0$. This gives, as $t \to i\infty$,

$$\theta(-1/2t) \sim C(-1/t)b_0/|W| . \tag{6.11}$$

The expansion (6.11) can be expressed as $t \to 0$

$$\theta(t) \sim a_0 t^{-(1/2)k} ,$$

with $a_0 = 2^{-(1/2)k} i^{-2n+(1/2)k} b_0/(|W| \operatorname{vol} P)$.

7. The volume of a Lie Group

In this section we shall prove Theorems 1.4 and 1.5. We start by recalling the result of the previous section. This is the asymptotic expansion

$$Z(-2\pi it) \sim e^{-2\pi iB(\rho)^2 t} a_0 t^{-(1/2)k}/|W| , \tag{7.1}$$

where the constant a_0 is given by

$$a_0 = i^{-3n+(1/2)k}(\Delta^n d^2)/[(4\pi i)^n n! \operatorname{vol} P] . \tag{7.2}$$

On the other hand from [7] we have the expansion

$$Z(-2\pi it) \sim (-8\pi^2 it)^{-(1/2)k} \operatorname{vol} G(1 + O(t)) . \tag{7.3}$$

Equating the first terms of these expansions leads after some elementary manipulation to the result

$$\operatorname{vol} G = 2^l \pi^{n+l}(\Delta^n d^2)/(n! \, |W| \operatorname{vol} P) . \tag{7.4}$$

We shall use this to obtain a simple expression for the volume of the Lie group G. First we need to calculate $\Delta^n d^2$.

Let $\tau = (-1)^n \Sigma B(\alpha_1, \alpha_2) \cdots B(\alpha_{2n-1}, \alpha_{2n})$ where the summation is over all $(2n)!$ orderings $(\alpha_1, \cdots, \alpha_{2n})$ of the roots of G with respect to the fixed maximal torus T. The value of $\Delta^n d^2$ is then given by

Lemma 7.1. $\Delta^n d^2 = \tau / \prod_{\alpha>0} B(\alpha, \rho)^2$.

Proof. Rather than calculate $\Delta^n d^2$ we shall calculate $\Delta^n(\Pi\alpha)$, where the product is taken over all the roots. To see that this is sufficient observe the following. Let $\omega: \mathfrak{t}^* \to \mathfrak{t}$ be the isomorphism induced by B so that $\lambda(\omega(\mu)) = B(\lambda, \mu)$ for all $\lambda, \mu \in \mathfrak{t}^*$, and let $d^* = d \circ \omega^{-1}$ then since ω is an isometry $\Delta^n d^2 = \Delta^n d^{*2}$, where Δ denotes respectively the Laplacian associated to B on either $\mathfrak{t}^*$ or $\mathfrak{t}$. Now $d^* = \prod_{\alpha>0} \alpha / \prod_{\alpha>0} B(\alpha, \rho)$, hence we have

$$\Delta^n d^2 = (-1)^n \Delta^n(\Pi\alpha)/ \prod_{\alpha>0} B(\alpha, \rho)^2 , \tag{7.5}$$

where the first product is taken over all the roots. Thus the proof of the lemma will be complete when we show $\tau = (-1)^n \Delta^n(\Pi\alpha)$. To see this let $x_1, \cdots, x_l$ be an orthonormal basis of t with respect to $B(x, y)$, and let $\xi_1, \cdots, \xi_l$ be the dual basis of t^*. Now $\alpha = \Sigma\alpha(x_j)\xi_j$ and so $\partial\alpha/\partial\xi_j = \alpha(x_j)$. Hence we have

$$\frac{\partial^2}{\partial\xi_j^2}(\Pi\alpha) = \sum_{\alpha\neq\beta}\alpha(x_j)\beta(x_j)\prod_{\gamma\neq\alpha,\beta}\gamma\,, \tag{7.6}$$

where now all the sums and products are over all the roots with the indicated restrictions. From (7.6) and the definition of $\{x_j\}$ we have

$$\Delta(\Pi\alpha) = \sum_{\alpha\neq\beta} B(\alpha, \beta)\prod_{\gamma\neq\alpha,\beta}\gamma\,,$$

and so

$$\Delta^n(\Pi\alpha) = \Sigma B(\alpha_1, \alpha_2)\cdots B(\alpha_{2n-1}, \alpha_{2n})\,,$$

where this summation is over all $(2n)!$ orderings $(\alpha_1, \cdots, \alpha_{2n})$ of the set of roots. It is now clear from the definitions that $\tau = (-1)^n\Delta^n(\Pi\alpha)$ which together with (7.5) completes the proof of the lemma.

If we combine Lemma 7.1 and (7.4) we obtain the expression

$$\mathrm{vol}\, G = 2^l\pi^{n+l}\tau/[\prod_{\alpha>0} B(\alpha, \rho)^2 n!\,|W|\,\mathrm{vol}\, P]\,. \tag{7.7}$$

There is a similar expression for vol G due to Freudenthal [3], which in our notation should read

$$\mathrm{vol}\, G = 2^{l+2n}\pi^{l+n}n!\,|W|\,\mathrm{vol}\, Q(R^v)/\tau\,. \tag{7.8}$$

We can obtain a value for the number τ by equating these two expressions. More precisely we find

$$\tau = 2^n n!\,|W|\prod_{\alpha>0} B(\alpha, \rho)\,, \tag{7.9}$$

where we have used the facts that vol P vol $Q(R^v) = 1$ and that τ is positive. This gives the volume of G as

$$\mathrm{vol}\, G = (2\pi)^{l+n}\,\mathrm{vol}\, Q(R^v)/\prod_{\alpha>0} B(\alpha, \rho)\,, \tag{7.10}$$

which is Theorem 1.4.

This formula (7.10) can be expressed in other forms, one of which is given in Theorem 1.5. To see this we need to introduce the height of a coroot α^v, for the definition of coroots see [2].

Definition. $\rho(\alpha^v)$ is the height of the coroot α^v, where $\rho \in \mathfrak{t}^*$ is half the sum of the positive roots and $\alpha^v \in \mathfrak{t}$. Some of the properties of $\rho(\alpha^v)$ are well known. First, $1 \leq \rho(\alpha^v) \leq h - 1$ where h is the Coxeter number. Let n_r be the number of coroots of height r. Then $(n_1, \cdots, n_{h-1})$ is a partition of n. Now let m_j be the number of n_r such that $n_r \geq j$. Then there are at most l nonzero m_j, $m_1 \geq m_2 \geq \cdots m_l$, which are the exponents of G. Thus we have

$$\prod_{\alpha>0} \rho(\alpha^v) = \prod_{j=1}^{l} (m_j!) ,$$

and hence

$$\prod_{\alpha>0} B(\alpha, \rho) = 2^n \prod_{j=1}^{l} (m_j!)/ \prod_{\alpha>0} B(\alpha^v)^2 . \tag{7.11}$$

With this we can write (7.10) as

$$\text{vol } G = 2^l \pi^{l+n} \prod_{\alpha>0} B(\alpha^v)^2 \text{ vol } Q(R^v)/ \prod (m_j!) . \tag{7.12}$$

Since the volume of the unit sphere $\boldsymbol{S}^{2m+1} \subset \boldsymbol{R}^{2m+2}$ with the standard Euclidean measure is vol $\boldsymbol{S}^{2m+1} = 2\pi^{m+1}/m!$, (7.12) can be written

$$\text{vol } G = \text{vol } Q(R^v) B(\alpha^v)^2 \prod \text{vol } \boldsymbol{S}^{2m_j+1} . \tag{7.13}$$

(7.13) is Theorem 1.5 for the group G. In fact it holds more generally. For any Lie group G, which is compact and connected, we fix a maximal torus T. Then the integer lattice is $L = (2\pi)^{-1} \ker (\exp: \mathfrak{t} \to T)$, and we denote the degrees of the generators of $H^*(G, \boldsymbol{R})$ by $\{d_j\}$.

If $\langle x, y \rangle$ is an Ad_G-invariant innerproduct on $\mathfrak{g}$, then the volume of G with respect to the Riemannian structure induced by $\langle x, y \rangle$ is

$$\text{vol } G = C \prod \text{vol } \boldsymbol{S}^{d_j} ,$$

and the constant C is given by

$$C = \text{vol } L \prod_{\alpha>0} \|\alpha^v\|^2 .$$

References

[1] M. Berger, *Geometry of the spectrum*. I, Proc. Sympos. Pure Math. Vol. 27, Pt. 2, Amer. Math. Soc. 1975, 129–152.

[2] N. Bourbaki, *Groupes et algébres de Lie*, chap. 4, 5 et 6, Hermann, Paris, 1968.

[3] H. Freudenthal & H. de Vries, *Linear Lie groups*, Academic Press, New York, 1969.

[4] N. Jacobson, *Lie algebras*, Interscience, New York, 1962.

[5] B. Kostant, *On Macdonald's η-function formula, the Laplacian and generalized exponents*, Advances in Math. **20** (1976) 179–212.

[6] I. G. Macdonald, *Affine root systems and Dedekind's η-function*, Invent. Math. **15** (1972) 91–143.

[7] H. P. McKean & I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Differential Geometry **1** (1967) 43–69.

[8] J. Milnor, *Morse theory*, Annals of Math. Studies, No. 51, Princeton University Press, Princeton, 1963.

[9] W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1966.

[10] H. Urakawa, *The heat equation on compact Lie group*, Osaka J. Math. **12** (1975) 285–297.

[11] B. Van Asch, *Des identités pour certain puissances de η*, C. R. Acad. Sci. Paris Sér. A, **277** (1973) 1087–1090.

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